2 Use these angles and equations 6.18 (p. 317) to find ω_3 and ω_4 for the open circuit.

$$\omega_3 = \frac{a\omega_2}{b}\frac{\sin(\theta_4-\theta_2)}{\sin(\theta_3-\theta_4)} = \frac{40(25)}{120}\frac{\sin(57.325°-40°)}{\sin(20.298°-57.325°)} = -4.121 \text{ rad/sec}$$

$$\omega_4 = \frac{a\omega_2}{c}\frac{\sin(\theta_2-\theta_3)}{\sin(\theta_4-\theta_3)} = \frac{40(25)}{80}\frac{\sin(40°-20.298°)}{\sin(57.325°-20.298°)} = 6.998 \text{ rad/sec} \tag{a}$$

3 Use the angular velocities and equations 6.19 (p. 317) to find the linear velocities of points A and B.

$$\begin{aligned}\mathbf{V}_A &= a\omega_2(-\sin\theta_2 + j\cos\theta_2)\\ &= 40(25)(-\sin 40° + j\cos 40°) = -642.79 + j766.04\end{aligned}$$

$$\mathbf{V}_{A_x} = -642.79; \quad \mathbf{V}_{A_y} = 766.04; \quad \mathbf{V}_{A_{mag}} = 1000 \text{ mm/sec}; \quad \mathbf{V}_{A_{ang}} = 130° \tag{b}$$

$$\begin{aligned}\mathbf{V}_{BA} &= b\omega_3(-\sin\theta_3 + j\cos\theta_3)\\ &= 120(-4.121)(-\sin 20.298° + j20.298°) = 171.55 - j463.80\end{aligned}$$

$$\mathbf{V}_{BA_x} = 171.55; \quad \mathbf{V}_{BA_y} = -463.80; \quad \mathbf{V}_{BA_{mag}} = 494.51 \text{ mm/sec}; \quad \mathbf{V}_{BA_{ang}} = -69.70° \tag{c}$$

$$\begin{aligned}\mathbf{V}_B &= c\omega_4(-\sin\theta_4 + j\cos\theta_4)\\ &= 80(6.998)(-\sin 57.325 + j\cos 57.325) = -471.242 + j302.243\end{aligned}$$

$$\mathbf{V}_{B_x} = -471.242; \quad \mathbf{V}_{B_y} = 302.243; \quad \mathbf{V}_{B_{mag}} = 599.84 \text{ mm/sec}; \quad \mathbf{V}_{B_{ang}} = 147.33° \tag{d}$$

4 As an exercise, repeat the above process to find the velocities for the crossed circuit of the linkage.

The Fourbar Crank-Slider

The position equations for the fourbar offset crank-slider linkage (inversion #1) were derived in Section 4.6 (p. 191). The linkage was shown in Figure 4-9 (p. 191) and is shown again in Figure 6-21a on which we also show an input angular velocity ω_2 applied to link 2. This ω_2 can be a time-varying input velocity. The vector loop equation 4.14 is repeated here for your convenience.

$$\mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \tag{4.14a}$$

$$a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \tag{4.14b}$$

Differentiate equation 4.14b with respect to time noting that a, b, c, θ_1, and θ_4 are constant but the length of link d varies with time in this inversion.

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{d} = 0 \tag{6.20a}$$

The term $\dot{d}$ is the linear velocity of the slider block. Equation 6.20a is the velocity difference equation 6.5 (p. 287) and can be written in that form.

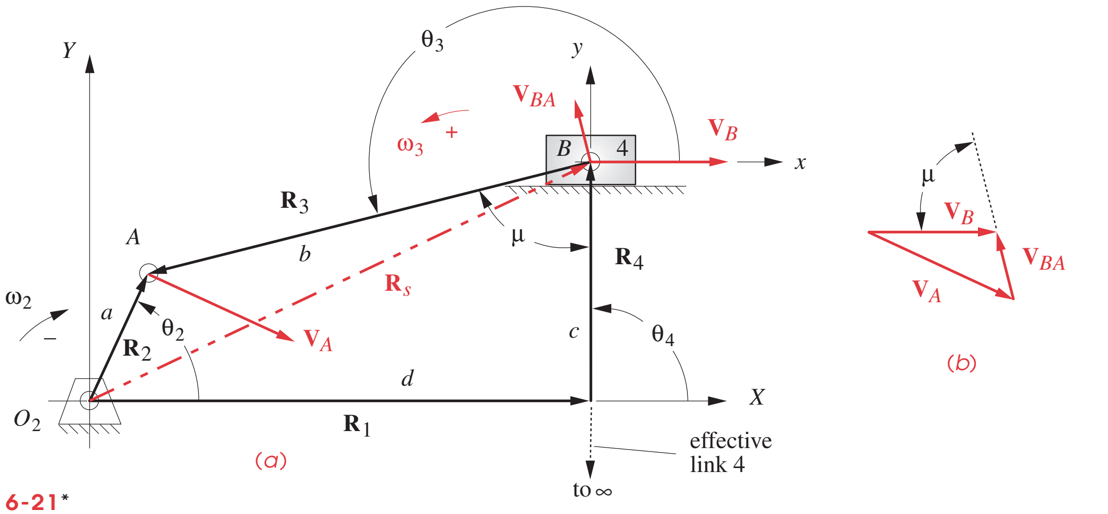

FIGURE 6-21*

Position vector loop for a fourbar slider-crank linkage showing velocity vectors for a negative (*CW*) ω_2

$$\mathbf{V}_A - \mathbf{V}_{AB} - \mathbf{V}_B = 0$$

or:
$$\mathbf{V}_A = \mathbf{V}_B + \mathbf{V}_{AB}$$

but:
$$\mathbf{V}_{AB} = -\mathbf{V}_{BA}$$

then:
$$\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA} \tag{6.20b}$$

Equation 6.20 is identical in form to equations 6.5 and 6.15a (p. 315). Note that because we arranged the position vector $\mathbf{R}_3$ in Figures 4-9 (p. 191) and 6-21 with its root at point B, directed from B to A, its derivative represents the velocity difference of point A with respect to point B, the opposite of that in the previous fourbar example. Compare this also to equation 6.15b noting that its vector $\mathbf{R}_3$ is directed from A to B. Figure 6-21b shows the vector diagram of the graphical solution to equation 6.20b.

Substitute the Euler equivalent, equation 4.4a (p. 185), in equation 6.20a,

$$ja\omega_2\left(\cos\theta_2 + j\sin\theta_2\right) - jb\omega_3\left(\cos\theta_3 + j\sin\theta_3\right) - \dot{d} = 0 \tag{6.21a}$$

simplify,

$$a\omega_2\left(-\sin\theta_2 + j\cos\theta_2\right) - b\omega_3\left(-\sin\theta_3 + j\cos\theta_3\right) - \dot{d} = 0 \tag{6.21b}$$

and separate into real and imaginary components.

real part (x component):

$$-a\omega_2\sin\theta_2 + b\omega_3\sin\theta_3 - \dot{d} = 0 \tag{6.21c}$$

imaginary part (y component):

$$a\omega_2\cos\theta_2 - b\omega_3\cos\theta_3 = 0 \tag{6.21d}$$

* Note the transmission angle μ in Figure 6-21a drawn between link 3 and effective link 4 as previously defined. It is also shown drawn between vectors $\mathbf{V}_B$ and $\mathbf{V}_{BA}$ in Figure 6-21b, indicating an alternate way to define the transmission angle as *the acute angle between the absolute velocity and velocity difference vectors* at a point such as B. This approach does not require construction of the slider's effective link 4 to determine the transmission angle.

These are two simultaneous equations in the two unknowns, $\dot{d}$ and ω_3. Equation 6.21d (p. 319) can be solved for ω_3 and substituted into 6.21c to find $\dot{d}$.

$$\omega_3 = \frac{a}{b}\frac{\cos\theta_2}{\cos\theta_3}\omega_2 \tag{6.22a}$$

$$\dot{d} = -a\,\omega_2 \sin\theta_2 + b\,\omega_3 \sin\theta_3 \tag{6.22b}$$

The absolute velocity of point A and the velocity difference of point A versus point B are found from equation 6.20:

$$\mathbf{V}_A = a\omega_2\left(-\sin\theta_2 + j\cos\theta_2\right) \tag{6.23a}$$

$$\mathbf{V}_{AB} = b\omega_3\left(-\sin\theta_3 + j\cos\theta_3\right) \tag{6.23b}$$

$$\mathbf{V}_{BA} = -\mathbf{V}_{AB} \tag{6.23c}$$

EXAMPLE 6-8

Velocity Analysis of a Fourbar Crank-Slider Linkage with the Vector Loop Method.

Problem: Given a fourbar crank-slider linkage with the link lengths $L_2 = a = 40$ mm, $L_3 = b = 120$ mm, *offset* $= c = -20$ mm. For $\theta_2 = 60°$ and $\omega_2 = -30$ rad/sec, find ω_3 and linear velocities of points A and B for the open circuit. Use the angles and positions found for the same linkage and its link 2 position in Example 4-2 (p. 193).

Solution: (See Figure 6-21, p. 319, for nomenclature.)

1 Example 4-2 found angle $\theta_3 = 152.91°$ and slider position $d = 126.84$ mm for the open circuit.

2 Using equation 6.22a and the data from step 1, calculate the coupler angular velocity ω_3.

$$\omega_3 = \frac{a}{b}\frac{\cos\theta_2}{\cos\theta_3}\omega_2 = \frac{40}{120}\frac{\cos 60°}{\cos 152.91°}(-30) = 5.616 \text{ rad/sec} \tag{a}$$

3 Using equation 6.22b and the data from steps 1 and 2, calculate the slider velocity $\dot{d}$.

$$\dot{d} = -a\,\omega_2 \sin\theta_2 + b\,\omega_3 \sin\theta_3 = -40(-30)\sin 60° + 120(5.616)\sin 152.91° = 1346 \text{ mm/sec} \tag{b}$$

4 Using equation 6.23 and the result from step 2, calculate the linear velocities V_A and V_{BA}.

$$\mathbf{V}_A = a\omega_2\left(-\sin\theta_2 + j\cos\theta_2\right) = 40(-30)\left(-\sin 60° + j\cos 60°\right) = 1039.23 - j600$$

$$\mathbf{V}_{A_x} = 1039.23; \quad \mathbf{V}_{A_y} = -600; \quad \mathbf{V}_{A_{mag}} = 1200 \text{ mm/sec}; \quad \mathbf{V}_{A_{ang}} = -30° \tag{c}$$

$$\mathbf{V}_{AB} = b\omega_3\left(-\sin\theta_3 + j\cos\theta_3\right)$$

$$\mathbf{V}_{AB} = 120(5.616)\left(-\sin 152.91° + j\cos 152.91°\right) = -306.86 - j600$$

$$\mathbf{V}_{BA} = -\mathbf{V}_{AB} = 306.86 + j600$$

$$\mathbf{V}_{BA_x} = 306.86; \quad \mathbf{V}_{BA_y} = 600; \quad \mathbf{V}_{BA_{mag}} = 673.92 \text{ mm/sec}; \quad \mathbf{V}_{BA_{ang}} = 62.91° \tag{d}$$

The Fourbar Slider-Crank

The *fourbar* ***slider-crank*** *linkage* has the same geometry as the *fourbar* ***crank-slider*** *linkage* that was analyzed in the previous section. The name change indicates that it will be driven with the slider as input and the crank as output. This is sometimes referred to as a "back-driven" crank-slider. We will use the term **slider-crank** to define it as slider-driven. This is a very commonly used linkage configuration. Every internal-combustion, piston engine has as many of these as it has cylinders. The vector loop is as shown in Figure 6-21 and the vector loop equation is identical to that of the crank-slider (equation 4.14a, p. 192). The derivation for θ_2 as a function of slider position d was done in Section 4-7 (p. 194). Now we want to solve for ω_2 as a function of slider velocity $\dot{d}$ and the known link lengths and angles.

We can start with equations 6.21c and d, which also apply to this linkage:

$$-a\omega_2 \sin\theta_2 + b\omega_3 \sin\theta_3 - \dot{d} = 0 \tag{6.21c}$$

$$a\omega_2 \cos\theta_2 - b\omega_3 \cos\theta_3 = 0 \tag{6.21d}$$

Solve equation 6.21d for ω_3 in terms of ω_2.

$$\omega_3 = \frac{a\omega_2 \cos\theta_2}{b\cos\theta_3} \tag{6.24a}$$

Substitute equation 6.24a for ω_3 in equation 6.21c and solve for ω_2.

$$\omega_2 = \frac{\dot{d}\cos\theta_3}{a\left(\cos\theta_2 \sin\theta_3 - \sin\theta_2 \cos\theta_3\right)} \tag{6.24b}$$

The circuit of the linkage depends on the value of d chosen and the angular velocities will be for the circuit represented by the values of θ_2 and θ_3 used from equation 4.21 (p. 195).*

EXAMPLE 6-9

Velocity Analysis of a Fourbar Slider-Crank Linkage with the Vector Loop Method.

Problem: Given a fourbar slider-crank linkage with the link lengths $L_2 = a = 40$ mm, $L_3 = b = 120$ mm, *offset* $= c = -20$ mm. For $d = 100$ mm and $\dot{d} = 1200$ mm/sec, find ω_2 and ω_3 for both branches of one circuit of the linkage. Use the angles found for the same linkage in Example 4-3 (p. 196).

Solution: (See Figure 6-21 for nomenclature.)

1 Example 4-3 found angles $\theta_{2_1} = 95.798°$, $\theta_{3_1} = 150.113°$ for branch 1 and $\theta_{2_2} = -118.418°$, $\theta_{3_2} = 187.267°$ for branch 2 of this linkage.

2 Using equation 6.24b and the data from step 1, calculate the crank angular velocity ω_{2_1}.

$$\omega_{2_1} = \frac{\dot{d}\cos\theta_{3_1}}{a\left(\cos\theta_{2_1} \sin\theta_{3_1} - \sin\theta_{2_1} \cos\theta_{3_1}\right)}$$

$$= \frac{1200\cos 150.113°}{40\left(\cos 95.798° \sin 150.113° - \sin 95.798° \cos 150.113°\right)} = -32.023 \text{ rad/sec} \tag{a}$$

* The crank-slider and slider-crank linkage both have two circuits or configurations in which they can be independently assembled, sometimes called open and crossed. Because effective link 4 is always perpendicular to the slider axis, it is parallel to itself on both circuits. This results in the two circuits being mirror images of one another, mirrored about a line through the crank pivot and perpendicular to the slide axis. Thus, the choice of value of slider position d in the calculation of the slider-crank linkage determines which circuit is being analyzed. But, because of the change points at TDC and BDC, the slider crank has two branches on each circuit and the two solutions obtained from equation 4.21 represent the two branches on the one circuit being analyzed. In contrast, the crank-slider has only one branch per circuit because when the crank is driven, it can make a full revolution and there are no change points to separate branches. See Section 4.13 (p. 208) for a more complete discussion of circuits and branches in linkages.

3 Using equation 6.24a (p. 321) and data from steps 1 and 2, calculate coupler angular velocity ω_{3_1}.

$$\omega_{3_1} = \frac{a\omega_{2_1}\cos\theta_{2_1}}{b\cos\theta_{3_1}} = \frac{40(-32.023)\cos 95.798°}{120\cos 150.113°} = -1.244 \text{ rad/sec} \tag{b}$$

4 Example 4-3 (p. 196) found $\theta_{2_2} = -118.418°$ and $\theta_{3_2} = 187.267°$ for branch 2 of this linkage.

5 Using equation 6.24b and the data from step 2, calculate the crank angular velocity ω_{2_2}.

$$\omega_{2_2} = \frac{\dot{d}\cos\theta_{3_2}}{a\left(\cos\theta_{2_2}\sin\theta_{3_2} - \sin\theta_{2_2}\cos\theta_{3_2}\right)}$$

$$= \frac{1200\cos(187.267°)}{40\left[\cos(-118.418°)\sin(187.267°) - \sin(-118.418°)\cos(187.267°)\right]} = 36.639 \text{ rad/sec} \tag{c}$$

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6 Using equation 6.24a and the data from steps 3 and 4, calculate coupler angular velocity ω_{3_2}.

$$\omega_{3_2} = \frac{a\omega_{2_2}\cos\theta_{2_2}}{b\cos\theta_{3_2}} = \frac{40(36.639)\cos(-118.418°)}{120\cos(187.267°)} = 5.859 \text{ rad/sec} \tag{d}$$

The Fourbar Inverted Crank-Slider

The position equations for the fourbar inverted crank-slider linkage were derived in Section 4.7 (p. 194). The linkage was shown in Figure 4-10 (p. 192) and is shown again in Figure 6-22 on which we also show an input angular velocity ω_2 applied to link 2. This ω_2 can vary with time. The vector loop equations 4.14 repeated on p. 318 are valid for this linkage as well.

All slider linkages will have at least one link whose effective length between joints varies as the linkage moves. In this inversion the length of link 3 between points *A* and *B*, designated as *b*, will change as it passes through the slider block on link 4. To get an expression for velocity, differentiate equation 4.14b with respect to time noting that *a*, *c*, *d*, and θ_1 are constant and *b* varies with time.

$$ja\omega_2 e^{j\theta_2} - jb\omega_3 e^{j\theta_3} - \dot{b}e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0 \tag{6.25a}$$

The value of *db*/*dt* will be one of the variables to be solved for in this case and is the $\dot{b}$ term in the equation. Another variable will be ω_4, the angular velocity of link 4. Note, however, that we also have an unknown in ω_3, the angular velocity of link 3. This is a total of three unknowns. Equation 6.25a can only be solved for two unknowns. Thus we require another equation to solve the system. There is a fixed relationship between angles θ_3 and θ_4, shown as γ in Figure 6-22 and defined in equation 4.22, repeated here:

$$\theta_3 = \theta_4 + \gamma \tag{4.22}$$

Differentiate it with respect to time to obtain:

$$\omega_3 = \omega_4 \tag{6.25b}$$

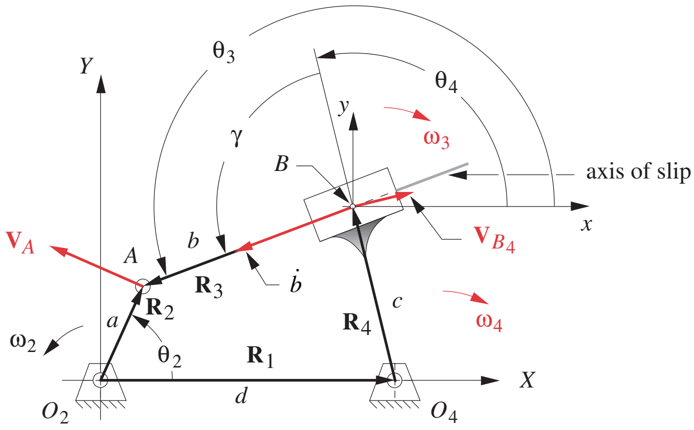

FIGURE 6-22

Velocity analysis of inversion #3 of the slider-crank fourbar linkage

We wish to solve equation 6.25a to get expressions in this form:

$$\begin{aligned} \omega_3 = \omega_4 &= f(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2) \\ \frac{db}{dt} = \dot{b} &= g(a, b, c, d, \theta_2, \theta_3, \theta_4, \omega_2) \end{aligned} \tag{6.26}$$

Substitution of the Euler identity (equation 4.4a, p. 185) into equation 6.25a yields:

$$\begin{aligned} &ja\omega_2(\cos\theta_2 + j\sin\theta_2) - jb\omega_3(\cos\theta_3 + j\sin\theta_3) \\ &\quad -\dot{b}(\cos\theta_3 + j\sin\theta_3) - jc\omega_4(\cos\theta_4 + j\sin\theta_4) = 0 \end{aligned} \tag{6.27a}$$

Multiply by the operator j and substitute ω_4 for ω_3 from equation 6.25b:

$$\begin{aligned} &a\omega_2(-\sin\theta_2 + j\cos\theta_2) - b\omega_4(-\sin\theta_3 + j\cos\theta_3) \\ &\quad -\dot{b}(\cos\theta_3 + j\sin\theta_3) - c\omega_4(-\sin\theta_4 + j\cos\theta_4) = 0 \end{aligned} \tag{6.27b}$$

We can now separate this vector equation into its two components by collecting all real and all imaginary terms separately:

real part (x component):

$$-a\omega_2\sin\theta_2 + b\omega_4\sin\theta_3 - \dot{b}\cos\theta_3 + c\omega_4\sin\theta_4 = 0 \tag{6.28a}$$

imaginary part (y component):

$$a\omega_2\cos\theta_2 - b\omega_4\cos\theta_3 - \dot{b}\sin\theta_3 - c\omega_4\cos\theta_4 = 0 \tag{6.28b}$$

Collect terms and rearrange equations 6.28 to isolate one unknown on the left side.

$$\dot{b}\cos\theta_3 = -a\omega_2\sin\theta_2 + \omega_4(b\sin\theta_3 + c\sin\theta_4) \tag{6.29a}$$

$$\dot{b}\sin\theta_3 = a\omega_2\cos\theta_2 - \omega_4(b\cos\theta_3 + c\cos\theta_4) \tag{6.29b}$$

Either equation can be solved for $\dot{b}$ and the result substituted in the other. Solving equation 6.29a (p. 323):

$$\dot{b} = \frac{-a\omega_2 \sin\theta_2 + \omega_4 \left(b \sin\theta_3 + c \sin\theta_4\right)}{\cos\theta_3} \tag{6.30a}$$

Substitute in equation 6.29b (p. 323) and simplify:

$$\omega_4 = \frac{a\omega_2 \cos\left(\theta_2 - \theta_3\right)}{b + c\cos\left(\theta_4 - \theta_3\right)} \tag{6.30b}$$

Equation 6.30a provides the **velocity of slip** at point B. Equation 6.30b gives the **angular velocity** of link 4. Note that we can substitute $-\gamma = \theta_4 - \theta_3$ from equation 4.18 (for an open linkage) into equation 6.30b to further simplify it. Note that $\cos(-\gamma) = \cos(\gamma)$.

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$$\omega_4 = \frac{a\omega_2 \cos\left(\theta_2 - \theta_3\right)}{b + c\cos\gamma} \tag{6.30c}$$

The **velocity of slip** from equation 6.30a is always directed along the **axis of slip** as shown in Figure 6-22 (p. 323). There is also a component orthogonal to the axis of slip called the **velocity of transmission**. This lies along the **axis of transmission** which is the only line along which any useful work can be transmitted across the sliding joint. All energy associated with motion along the slip axis is converted to heat and lost.

The absolute linear velocity of point A is found from equation 6.23a (p. 320). We can find the absolute velocity of point B on link 4 since ω_4 is now known. From equation 6.15b (p. 315):

$$\mathbf{V}_{B_4} = jc\omega_4 e^{j\theta_4} = c\omega_4\left(-\sin\theta_4 + j\cos\theta_4\right) \tag{6.31a}$$

The velocity of transmission is the component of V_{b4} normal to the axis of slip. The absolute velocity of point B on link 3 is found from equation 6.5 (p. 287) as

$$\mathbf{V}_{B_3} = \mathbf{V}_{B_4} + \mathbf{V}_{B_{34}} = \mathbf{V}_{B_4} + \mathbf{V}_{slip_{34}} \tag{6.31b}$$

6.8 VELOCITY ANALYSIS OF THE GEARED FIVEBAR LINKAGE

The position loop equation for the geared fivebar mechanism was derived in Section 4.8 (p. 197) and is repeated here. See Figure P6-4 (p. 331) for notation.

$$a\,e^{j\theta_2} + b\,e^{j\theta_3} - c\,e^{j\theta_4} - d\,e^{j\theta_5} - f\,e^{j\theta_1} = 0 \tag{4.27b}$$

Differentiate this with respect to time to get an expression for velocity.

$$a\omega_2 je^{j\theta_2} + b\omega_3 je^{j\theta_3} - c\omega_4 je^{j\theta_4} - d\omega_5 je^{j\theta_5} = 0 \tag{6.32a}$$

Substitute the Euler equivalents:

$$\begin{aligned} &a\omega_2 j\left(\cos\theta_2 + j\sin\theta_2\right) + b\omega_3 j\left(\cos\theta_3 + j\sin\theta_3\right) \\ &\quad -c\omega_4 j\left(\cos\theta_4 + j\sin\theta_4\right) - d\omega_5 j\left(\cos\theta_5 + j\sin\theta_5\right) = 0 \end{aligned} \tag{6.32b}$$