Table 10-1 Comparison of Shaft Design Results from Examples 10-1 and 10-2
Minimum Diameters Give $N_f = 2.5$ at Each Point

Design	Max Alt Torque	Max Mean Torque	Max Alt Moment	Max Mean Moment	d_0 (in) nom	d_1 (in) min / nom	d_2 (in) min / nom	d_3 (in) min / nom	S_f at C
Ex. 10-1	0	73.1	63.9	0	0.750	0.517 / 0.625	0.557 / 0.591	0.411 / 0.500	2.5 / 3.0
Ex. 10-2	73.1	73.1	63.9	63.9	0.875	0.632 / 0.750	0.614 / 0.669	0.512 / 0.531	2.5 / 3.8

Shafts as Beams

The methods of Section 4.10 are directly applicable. The only complicating factor is the usual presence of steps in a shaft that change the cross-sectional properties along its length. The integration of the M / EI function becomes much more complicated due to the fact that both I and M are now functions of the dimension along the shaft-beam. Rather than do an analytical integration as was done in Section 4.10 for the case of constant I, we will use a numerical integration technique such as Simpson's rule or the trapezoidal rule to form the slope and deflection functions from the M / EI function. This will be demonstrated in an example. If the transverse loads and moment are time varying, then the absolute maximum magnitudes should be used to calculate the deflections. The deflection function will depend on the loading and the beam boundary conditions, i.e., whether simply supported, cantilevered, or overhung.

Shafts as Torsion Bars

The methods of Section 4.12 are directly applicable, particularly equation 4.24 (p. 178), since the only practical shaft cross section is circular. The angular deflection θ (in radians) for a shaft of length l, shear modulus G, polar moment of inertia J, with torque T is

$$\theta = \frac{T\,l}{GJ} \tag{10.9a}$$

from which we can form the expression for the torsional spring constant:

$$k_t = \frac{T}{\theta} = \frac{GJ}{l} \tag{10.9b}$$

If the shaft is stepped, the changing cross sections complicate the torsional deflection and spring constant calculation due to the changing polar moment of inertia J.

Any collection of adjacent, different-diameter sections of shaft can be considered as a set of springs in series since their deflections add and the torque passes through unchanged. An effective spring constant or an effective J can be computed for any segment of shaft in order to find the relative deflection between its ends. For a segment of a shaft containing three sections of differing cross sections J_1, J_2, and J_3 with corresponding lengths l_1, l_2, l_3, the total deflection is merely the sum of the deflections of each section subjected to the same torque. We assume that the material is consistent throughout.

$$\theta = \theta_1 + \theta_2 + \theta_3 = \frac{T}{G}\left(\frac{l_1}{J_1} + \frac{l_2}{J_2} + \frac{l_3}{J_3}\right) \tag{10.9c}$$

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The effective spring constant k_{eff} of a three-segment stepped shaft is

$$\frac{1}{k_{t_{eff}}} = \frac{1}{k_{t_1}} + \frac{1}{k_{t_2}} + \frac{1}{k_{t_3}} \tag{10.9d}$$

These expressions can be extended to any number of segments of a stepped shaft.

EXAMPLE 10-3

Designing a Stepped Shaft to Minimize Deflection

Problem **Design the same shaft as in Example 10-2 to have a maximum bending deflection of 0.002 in and a maximum angular deflection of 0.5° between sheave and gear.**

Given **The loading is the same as in Example 10-2. The peak torque is 146 lb-in. Figure 10-9 shows the distribution of the peak moment over the shaft length. The values are 65.6 lb-in at point *B*, 127.9 lb-in at point *C*, and 18.3 lb-in at point *D*.**

Assumptions **The lengths will remain the same as in previous example, but diameters can be changed to stiffen the shaft if necessary. The material is the same as in Example 10-2.**

Solution **See Figures 10-5 (p. 561), and 10-11 to 10-13.**

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1 The torsional deflection is found from equations 10.9. The lengths of each segment are (from Figure 10-5 on p. 561): AB = 1.5 in, BC = 3.5 in, and CD = 1.5 in. The polar area moments of inertia are first calculated for each segment of different diameter.

$$\begin{aligned}
&\text{from } A \text{ to } B: \quad J_{AB} = \frac{\pi d_{AB}^4}{32} = \frac{\pi(0.875)^4}{32} = 0.057\,5 \text{ in}^4 \\
&\text{from } B \text{ to } C: \quad J_{BC} = \frac{\pi d_{BC}^4}{32} = \frac{\pi(0.750)^4}{32} = 0.031\,1 \text{ in}^4 \\
&\text{from } C \text{ to } D: \quad J_{CD} = \frac{\pi d_{CD}^4}{32} = \frac{\pi(0.669)^4}{32} = 0.019\,7 \text{ in}^4
\end{aligned} \tag{a}$$

and used in equation 10.9c (p. 567).

$$\begin{aligned}
\theta &= \frac{T}{G}\left(\frac{l_1}{J_1} + \frac{l_2}{J_2} + \frac{l_3}{J_3}\right) \\
&= \frac{146}{1.2E7}\left(\frac{1.5}{0.057\,5} + \frac{3.5}{0.031\,1} + \frac{1.5}{0.019\,7}\right) = 0.15 \text{ deg}
\end{aligned} \tag{b}$$

This deflection is within the requested specification.

2 The moment function for this shaft was derived using singularity functions as equation *(j)* in Example 10-1 (p. 562). It must now be divided by the product of E and the area moment of inertia I at each point along the shaft axis. While E is constant, the value of I changes with each diametral change in the stepped shaft so we need to create a function for the variable I and singularity functions will do this.

$$\frac{M}{EI} = \frac{M}{E\left[I_{AB}\langle z-0\rangle^0 + \left(I_{BC}-I_{AB}\right)\langle z-1.5\rangle^0 + \left(I_{CD}-I_{BC}\right)\langle z-5\rangle^0 + \left(I_D-I_{CD}\right)\langle z-6.5\rangle^0\right]} \quad (c)$$

where I_{AB}, I_{BC}, I_{CD}, and I_D are the area moments of inertia of the respective step diameters on the shaft and are equal to half of the respective J values in equation (*a*).

Figure 10-11*a* shows the moment function for this shaft as derived in the previous examples, Figure 10-11*b* shows the *EI* function for the section diameters defined in Example 10-2 from equation (*c*) and Figure 10-11*c* shows the *M* / *EI* function.

3 The bending deflection is found by integrating the *M* / *EI* function twice.

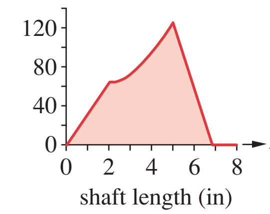
$$\theta = \int \frac{M}{EI} dz + C_3 \quad (d)$$

$$\delta = \iint \frac{M}{EI} dz + C_3 z + C_4 \quad (e)$$

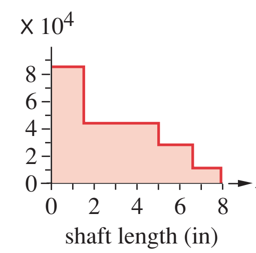
4 The first integration of the *M* / *EI* function from equation (*c*) gives the beam slope and the second integration gives the deflection function. In previous discussions of beam deflection (see Section 4.10 on p. 162 and Examples 4-4 to 4-7 on pp. 164 to 173) the cross section *I* of the beam was constant across its length. In a stepped shaft, *I* is a function of the shaft length. This makes the analytical integration of the *M* / *EI* function much more complicated. A simpler approach is to numerically integrate the function twice using a trapezoidal or Simpson's rule. This numerical integration must be done for each coordinate direction to obtain the *x* and *y* components of deflection. These are then combined vectorially to get the deflection-magnitude and phase-angle functions over the shaft length.

5 Since the shaft deflection is zero at $z = 0$, $C_4 = 0$. The other constant of integration C_3 can be determined numerically. Figure 10-12*a* shows the beam slope in the *y* direction as integrated by a trapezoidal rule, and also shows the corrected slope function. The integrated result is shifted up by the integration constant C_3. However, we do not know where the proper zero crossover is for this function, so we cannot determine C_3 from the beam-slope function.

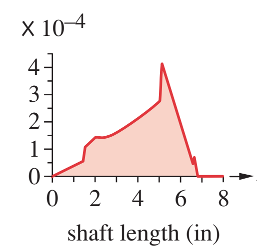
6 The as-integrated deflection function in Figure 10-12*b* does not equal zero at the second support. Since the deflection is really zero there, the error in this integrated



(*a*) Moment magnitude



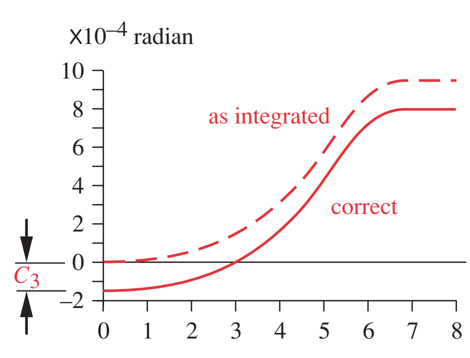
(*b*) EI function across shaft



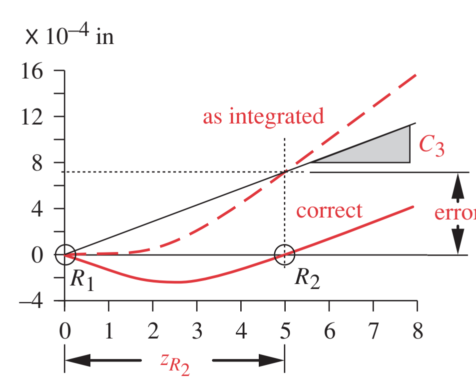
(*c*) Moment / *E I*

FIGURE 10-11

Moment and Moment / *EI* Functions in Example 10-3



(*a*) Beam slope



(*b*) Beam deflection

FIGURE 10-12

Numerical Integration of Moment Function and Finding the Integration Constant C_3