Clearly we can drop all terms after the second with very small error. Substituting this approximate expression for the radical in equation 13.1d (p. 668) gives an approximate expression for piston displacement with only a fraction of one percent error.

$$x = r\cos\omega t + l\left[1 - \left(\frac{r^2}{2l^2}\right)\sin^2\omega t\right] \tag{13.3a}$$

Substitute the trigonometric identity:

$$\sin^2\omega t = \frac{1-\cos 2\omega t}{2} \tag{13.3b}$$

and simplify:

$$x = l - \frac{r^2}{4l} + r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \tag{13.3c}$$

Differentiate for velocity of the piston (with constant ω):

$$\dot{x} \cong -r\omega\left(\sin\omega t + \frac{r}{2l}\sin 2\omega t\right) \tag{13.3d}$$

Differentiate again for acceleration (with constant ω):

$$\ddot{x} \cong -r\omega^2\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) \tag{13.3e}$$

The process of binomial expansion has, in this particular case, led us to Fourier series approximations of the exact expressions for the piston displacement, velocity, and acceleration. Fourier* showed that any periodic function can be approximated by a series of sine and cosine terms of integer multiples of the independent variable. Recall that we dropped the fourth, sixth, and subsequent power terms from the binomial expansion, which would have provided cos 4ω*t*, cos 6ω*t*, etc., terms in this expression. These multiple-angle functions are referred to as the **harmonics** of the fundamental cos ω*t* term. The cos ω*t* term repeats once per crank revolution and is called the fundamental frequency or the **primary component**. The second harmonic, cos 2ω*t*, repeats twice per crank revolution and is called the **secondary component**. The higher harmonics were dropped when we truncated the series. The constant term in the displacement function is the **DC component** or **average value**. The complete function is the sum of its harmonics. The Fourier series form of the expressions for displacement and its derivatives lets us see the relative contributions of the various harmonic components of the functions. This approach will prove to be quite valuable when we attempt to dynamically balance an engine design.

Program ENGINE calculates the position, velocity, and acceleration of the piston according to equations 13.3c, d, and e. Figure 13-8a, b, and c (p. 672) shows these functions for this example engine in the program as plotted for constant crank ω over two full revolutions. The acceleration curve shows the effects of the second harmonic term most clearly because that term's coefficient is larger than its correspondent in either of the other two functions. The fundamental (–cos ω*t*) term gives a pure harmonic function with a period of 360°. This fundamental term dominates the function as it has the largest coefficient in equation 13.3e. The flat top and slight dip in the positive peak acceleration

* Baron Jean Baptiste Joseph Fourier (1768-1830) published the description of the mathematical series which bears his name in *The Analytic Theory of Heat* in 1822. The Fourier series is widely used in harmonic analysis of all types of physical systems. Its general form is:

$$y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + \cdots + (a_n\cos nx + b_n\sin nx)$$